Mécanique Quantique I

Séance d'exercices n°1

1.

a) $[\lambda\hat{A}+\mu\hat{B},\hat{C}] = -[\hat{C},\lambda\hat{A}+\mu\hat{B}] = -[\hat{C},\lambda\hat{A}] - [\hat{C},\mu\hat{B}]$
$= -\lambda[\hat{C},\hat{A}] - \mu[\hat{C},\hat{B}] = \lambda[\hat{A},\hat{C}] + \mu[\hat{B},\hat{C}]$

b) $[\hat{A},\hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{A}\hat{C}$
$= \hat{B}[\hat{A},\hat{C}] + [\hat{A},\hat{B}]C$

c) $[\hat{A}\hat{B},\hat{C}] = -[\hat{C},\hat{A}\hat{B}] = -\hat{A}[\hat{C},\hat{B}] - [\hat{C},\hat{A}]\hat{B} = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$

d) $[\hat{A},[\hat{B},\hat{C}]] + [\hat{B},[\hat{C},\hat{A}]] + [\hat{C},[\hat{A},\hat{B}]] =$
$= \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} - \hat{B}\hat{C}\hat{A} + \hat{C}\hat{B}\hat{A}$
$+ \hat{B}\hat{C}\hat{A} - \hat{B}\hat{A}\hat{C} - \hat{C}\hat{A}\hat{B} + \hat{A}\hat{C}\hat{B}$
$+ \hat{C}\hat{A}\hat{B} - \hat{C}\hat{B}\hat{A} - \hat{A}\hat{B}\hat{C} + \hat{B}\hat{A}\hat{C} = 0$

2.

$$\left.\begin{array}{l}\text{Ex. 1, (d)}: [\hat{A},[\hat{B},\hat{C}]] + [\hat{B},[\hat{C},\hat{A}]] + [\hat{C},[\hat{A},\hat{B}]] = 0 \\ \text{Si } [A,\hat{B}] = 0 \text{ et } [\hat{B},\hat{C}] = 0\end{array}\right\} \Rightarrow [\hat{B},[\hat{C},\hat{A}]] = 0$$

Mais on peut pas déduire que $[\hat{C},\hat{A}] = 0$

$$\text{Ou}: \left.\begin{array}{l}[\hat{C}\hat{A},\hat{B}] = \hat{C}[\hat{A},\hat{B}] - [\hat{C},\hat{B}]\hat{A} \\ [\hat{A},\hat{B}] = 0 \\ [\hat{C},\hat{B}] = 0\end{array}\right\} \Rightarrow [\hat{C}\hat{A},\hat{B}] = 0 \Rightarrow \hat{C}\hat{A}\hat{B} - \hat{C}\hat{B}\hat{A} = 0$$

mais pas $\hat{C}\hat{A} = \hat{A}\hat{C}$

3.

a) $|\phi_1\rangle = a|v_1\rangle + ib|v_2\rangle \quad , |\phi_2\rangle = a|v_1\rangle - ib|v_2\rangle$

$\text{Norme}_1 = \sqrt{\langle\phi_1|\phi_1\rangle}$, $\text{Norme}_2 = \sqrt{\langle\phi_2|\phi_2\rangle}$, Produit Scalaire : $\langle\phi_1|\phi_2\rangle$

Vecteurs normalisés : $|\tilde{\phi}_1\rangle$, $|\tilde{\phi}_2\rangle$

$\langle\phi_1|\phi_1\rangle = (a\langle v_1| - ib\langle v_2|)(a|v_1\rangle + ib|v_2\rangle)$
$= a^2\|N_{v_1}\|^2 + 0 - 0 + b^2\|N_{v_2}\|^2$
$= a^2\|N_{v_1}\|^2 + b^2\|N_{v_2}\|^2$

$|v_1\rangle, |v_2\rangle$ non-normés
$\Rightarrow \langle v_1|v_1\rangle = \|N_{v_1}\|^2$
et $\langle v_2|v_2\rangle = \|N_{v_2}\|^2$

$\widetilde{|v_1\rangle, |v_2\rangle}$ orthogonaux
$\Rightarrow \langle v_1|v_2\rangle = 0$
et $\langle v_2|v_1\rangle = 0$

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$$\langle \phi_2|\phi_2\rangle = \left(a\langle v_1| + ib\langle v_2|\right)\left(a|v_1\rangle - ib|v_2\rangle\right)$$
$$= a^2\|N_{v_1}\|^2 - 0 + 0 + b^2\|N_{v_2}\|^2$$
$$= a^2\|N_{v_1}\|^2 + b^2\|N_{v_2}\|^2$$

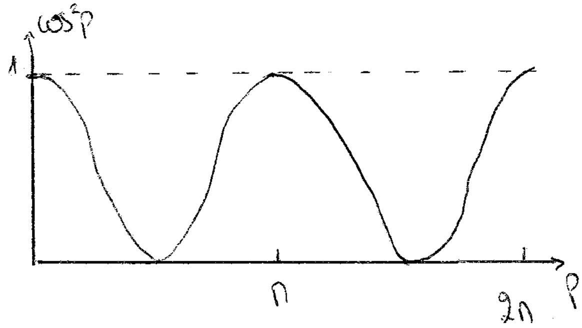
$$\langle \phi_1|\phi_2\rangle = \left(a\langle v_1| - ib\langle v_2|\right)\left(a|v_1\rangle - ib|v_2\rangle\right)$$
$$= a^2\|N_{v_1}\|^2 - 0 - 0 - b^2\|N_{v_2}\|^2$$
$$= a^2\|N_{v_1}\|^2 - b^2\|N_{v_2}\|^2$$

$$|\tilde{\phi}_1\rangle = \frac{1}{\sqrt{\langle\phi_1|\phi_1\rangle}}|\phi_1\rangle = \frac{1}{\sqrt{a^2\|N_{v_1}\|^2 + b^2\|N_{v_2}\|^2}}|\phi_1\rangle$$

$$|\tilde{\phi}_2\rangle = \frac{1}{\sqrt{\langle\phi_2|\phi_2\rangle}}|\phi_2\rangle = \frac{1}{\sqrt{a^2\|N_{v_1}\|^2 + b^2\|N_{v_2}\|^2}}|\phi_2\rangle$$

b)

$$\langle\phi_1|\phi_1\rangle = \frac{2}{\pi}\int_\alpha^{\alpha+\pi}\int_\alpha^{\alpha+\pi} dp\, dq\, \underbrace{\langle v_q|v_p\rangle}_{\delta(p-q)} \cos p \cdot \cos q$$



$$= \frac{2}{\pi}\int_\alpha^{\alpha+\pi} dp \cos^2 p$$

$$= \frac{\cancel{2}}{\pi}\int_\alpha^{\alpha+\pi} \frac{1+\cos 2p}{\cancel{2}}\, dp$$

$$= \frac{1}{\pi}\int_\alpha^{\alpha+\pi}(1+\cos 2p)\, dp = \frac{1}{\pi}\left[p\Big|_\alpha^{\alpha+\pi} + \int_\alpha^{\alpha+\pi}\cos 2p\, dp\right]$$

$$= \frac{1}{\cancel{\pi}}(\cancel{\alpha} + \cancel{\pi} - \cancel{\alpha}) + \frac{1}{\pi}\left(-\frac{\sin 2p}{2}\right)\Big|_\alpha^{\alpha+\pi}$$

$$= 1 - \frac{1}{2\pi}\left(\sin 2(\alpha+\pi) - \sin 2\alpha\right)$$

$$= 1 - \frac{1}{2\pi}\left(\sin(2\alpha+2\pi) - \sin 2\alpha\right)$$

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3.

b) $\langle \varphi_2 | \varphi_2 \rangle = \frac{2}{n} \int_\alpha^{\alpha+n} \int_\alpha^{\alpha+n} dp\, dq \underbrace{\langle v_q | v_p \rangle}_{\delta(p-q)} \sin p \sin q$

$= \frac{2}{n} \int_\alpha^{\alpha+n} dp \sin^2 p = \frac{2}{n} \int_\alpha^{\alpha+n} \frac{1-\cos 2\alpha}{2} \cdot dp$

$= \dots = 1$

4.

Si $\hat{A}^+ = \hat{A} \Rightarrow \hat{A}$: Hermitien

a) $\hat{A}^+ = \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \hat{A} \Rightarrow \hat{A}$ hermitien

$\hat{B}^+ = \begin{pmatrix} 0 & -2i & 0 \\ 2i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{B} \Rightarrow \hat{B}$ hermitien

$\hat{C}^+ = \begin{pmatrix} 1 & -1 & 1 \\ -i & -i & -i \\ 1 & 1 & 1 \end{pmatrix} \neq \hat{C} \Rightarrow \hat{C}$ pas hermitien

b)

$\hat{A}$: une valeur propre est $\lambda = 1$

Pour les autres :

$\det(\hat{A} - \lambda I) = 0 \Rightarrow (1-\lambda)^2 - 1 = 0 \Rightarrow \lambda = 0,\ \lambda = 2$

$\lambda = 1$: $\hat{A}\varphi = \varphi \Rightarrow \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \left. \begin{array}{l} x + iy = x \Rightarrow y = 0 \\ -ix + y = y \overset{y=0}{\Rightarrow} x = 0 \\ z = z \quad , \ z : \text{n'importe} \end{array} \right\} \Rightarrow$

$\Rightarrow$ Le vecteur propre est $\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$

(on peut choisir $z = 1$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$)

$$\lambda=0: \hat{A}\psi=0 \Rightarrow \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0 \Rightarrow \begin{matrix} x+iy=0 \Rightarrow x=-iy \\ -ix+y=0 \\ z=0 \end{matrix}$$

$$\begin{pmatrix} x \\ ix \\ 0 \end{pmatrix} \text{ ou } \begin{pmatrix} -iy \\ y \\ 0 \end{pmatrix} \quad \begin{matrix} x=i \\ y=1 \end{matrix} \quad \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \text{ ou } \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$$

$\hat{B}: \lambda=0$

$dt(\hat{B}-\lambda I)=0 \Rightarrow \lambda^2-4=0 \Rightarrow \lambda=\pm 2$

$$\lambda=0: \hat{B}\psi=0 \Rightarrow \begin{pmatrix} 0 & -2i & 0 \\ 2i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0 \Rightarrow \left.\begin{matrix} -2iy=0 \Rightarrow y=0 \\ 2ix=0 \Rightarrow x=0 \end{matrix}\right\} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \text{ ou } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda=2: \hat{B}\psi=2\psi \Rightarrow \begin{pmatrix} 0 & -2i & 0 \\ 2i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \dots \Rightarrow \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \text{ ou } \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

$$\lambda=-2: \hat{B}\psi=-2\psi \Rightarrow \dots \Rightarrow \begin{matrix} x=iy \\ y=-ix \\ z=0 \end{matrix} \Rightarrow \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \text{ ou } \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$$

$\hat{C}$

$\det(\hat{C}-\lambda I)=0 \Rightarrow \dots \Rightarrow \lambda=0,\ \lambda=i,\ \lambda=2$

$$\lambda=0: \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} \qquad \lambda=i: \begin{pmatrix} 1 \\ 2i+1 \\ 1 \end{pmatrix} \qquad \lambda=2: \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

On remarque que les opérateurs $\hat{A}$ et $\hat{B}$ ont de vecteurs propres communes, alors $\hat{A}$ et $\hat{B}$ commutent.

$|A\alpha\rangle = \alpha|\alpha\rangle$ et $|B\alpha\rangle = b|\alpha\rangle$ | $|BA\alpha\rangle = \alpha|B\alpha\rangle = \alpha b|\alpha\rangle$ | $|AB\alpha\rangle = \alpha|B\alpha\rangle$, $|AB\alpha\rangle = ab|\alpha\rangle$

$\Rightarrow AB=BA \Rightarrow [A,B]=0$

c) $\hat{A}=\hat{A}^+$, $\langle\psi|A|\psi\rangle = \langle A^+\psi|\psi\rangle$

$\Rightarrow \langle\psi|\alpha\psi\rangle = \langle\alpha^*\psi|\psi\rangle$

$\Rightarrow \alpha=\alpha^* \Rightarrow \alpha\in\mathbb{R}$

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5.

$[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x}$

$\hat{x}\hat{p}\cdot f = x\left(-i\hbar\frac{\partial}{\partial x}\right)\cdot f = -xi\hbar\frac{\partial f}{\partial x}$

$\hat{p}\hat{x}\cdot f = -i\hbar\frac{\partial}{\partial x}(x\cdot f) = -i\hbar f - i\hbar x\frac{\partial f}{\partial x}$

$[\hat{x},\hat{p}] = \cancel{-xi\hbar\frac{\partial}{\partial x}} + i\hbar + \cancel{i\hbar x\frac{\partial}{\partial x}} = i\hbar$